Sum/Product - Rationals or Irrationals

MathBitsNotebook.com

Topical Outline | Algebra 1 Outline | MathBits' Teacher Resources

Terms of Use Contact Person: Donna Roberts

Statement: **"The sum of two rational numbers is rational."**

By definition, a rational number can be expressed as a fraction with integer values in the numerator and denominator (denominator not zero). So, adding two rationals is the same as adding two such fractions, which will result in another fraction of this same form since integers are closed under addition and multiplication. Thus, adding two rational numbers produces another rational number.

Proof:

Given: $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers with a, b, c, d integers$(b, d \neq 0)$.

Prove: $\frac{a}{b} + \frac{c}{b}$ or $\frac{a}{b} + \frac{c}{d}$ are rational numbers.

$$\left.\begin{array}{ll} \text{If denominators are the same:} & \frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \\ \text{If denominators are different:} & \frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd} \end{array}\right\} b \neq 0;\ d \neq 0$$

Since integers are closed under addition and multiplication, $a+c$, ad, cb, bd and $ad+cb$ are integers. Thus, $\frac{a+c}{b}$ and $\frac{ad+cb}{bd}$ are fractions with integer values in the numerators and denominators, making them rational numbers.

Statement: **"The product of two rational numbers is rational."**

Again, by definition, a rational number can be expressed as a fraction with integer values in the numerator and denominator (denominator not zero). So, multiplying two rationals is the same as multiplying two such fractions, which will result in another fraction of this same form since integers are closed under multiplication. Thus, multiplying two rational numbers produces another rational number.

Proof:

Given: $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers with a, b, c, d integers$(b, d \neq 0)$.

Prove: $\frac{a}{b} \times \frac{c}{d}$ is a rational number.

$$\left.\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \quad \right\} b \neq 0;\ d \neq 0$$

Since integers are closed under multiplication, ac and bd are integers.

Thus, $\frac{ac}{bd}$ is a fraction with integers in the numerator and denominator, making it a rational number.

Look out! This next part gets tricky!!

Statement: **"The sum of two irrational numbers is SOMETIMES irrational."**

The sum of two irrational numbers, in some cases, will be irrational. However, if the irrational parts of the numbers have a zero sum (cancel each other out), the sum will be rational.

$$4\sqrt{5} \div 3\sqrt{2} = 4\sqrt{5} + 3\sqrt{2} \text{ ; which is irrational}$$

$$(2 + 6\sqrt{7}) + (-6\sqrt{7}) = 2 \text{ ; which is rational}$$

Statement: **"The product of two irrational numbers is SOMETIMES irrational."**

The product of two irrational numbers, in some cases, will be irrational. However, it is possible that some irrational numbers may multiply to form a rational product.

$$\sqrt{5} \times \sqrt{2} = \sqrt{10} \text{ ; which is irrational}$$

$$\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4 \text{ ; which is rational}$$

MathBitsNotebook Algebra 1

Sum/Product - Rationals and Irrationals

MathBitsNotebook.com

Topical Outline | Algebra 1 Outline | MathBits' Teacher Resources

Terms of Use Contact Person: Donna Roberts

Statement: **"The sum of a rational number and an irrational number is irrational."**

By definition, an irrational number in decimal form goes on forever without repeating (a non-repeating, non-terminating decimal). By definition, a rational number in decimal form either terminates or repeats. If you add a non-repeating, non-terminating decimal to a terminating decimal, you still have a non-repeating, non-terminating decimal. If you add a non-repeating, non-terminating decimal to a repeating decimal, you will no longer have a repeating decimal.

Examples:

$$3+\sqrt{5}=3+2.236067977...=5.236067977... \text{ ; which is irrational}$$

$$4\frac{1}{3}+\pi=4.333333333...+3.141592653...=7.474925986... \text{ ; which is irrational}$$

Indirect Proof (Proof by Contradiction):

(Assume the opposite of what you want to prove, and show it leads to a contradiction of a known fact.)

Assume x is an irrational number, and the sum of x and a rational $\frac{a}{b}$ is a rational $\frac{c}{d}$, where a, b, c, and d are integers ($b,d \neq 0$). Then $x+\frac{a}{b}=\frac{c}{d}$. By subtraction, $x=\frac{c}{d}-\frac{a}{b}$, and $x=\frac{cb-ad}{bd}$. Since integers are closed under multiplication and subtraction, cb, ad, bd and $cb-ad$ are integers, making $\frac{cb-ad}{bd}$ a rational number by definition. This is a contradiction to the given fact that x is an irrational number. The assumption is wrong. The sum of a rational number and an irrational number is an irrational number.

BEWARE

This one is sneaky!!

Statement: **"The product of a rational number and an irrational number is SOMETIMES irrational."**

If you multiply any irrational number by the rational number zero, the result will be zero, which is rational. Any other situation, however, of a rational times an irrational will be irrational.

A better statement would be:

"The product of a non-zero rational number and an irrational number is irrational."

Indirect Proof (Proof by Contradiction) of the better statement:

(Assume the opposite of what you want to prove, and show it leads to a contradiction of a known fact.)

Assume x is an irrational number, and the product of x and a rational $\frac{a}{b}$ is a rational $\frac{c}{d}$, where a, b, c, and d are integers ($a,b,d \neq 0$). Then $x \cdot \frac{a}{b} = \frac{c}{d}$. By division, $x = \frac{c}{d} \div \frac{a}{b} = \frac{cb}{da}$. Since integers are closed under multiplication, cb, and da are integers, making $\frac{cb}{da}$ a rational number by definition. This is a contradiction to the given fact that x is an irrational number. The assumption is wrong. The product of a non-zero rational number and an irrational number is an irrational number.
